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An Efficient Technique for the Iterative Solution of Large Dense Matrices Arising in Method of Moments Simulation of MMIC Problems

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Abstract — **This paper presents a novel technique referred to herein as MNM for efficient iterative solution of MoM matrices over a broad frequency range. It utilizes a combination of techniques to reduce the number of iterations required to generate the solution including a special choice of the initial guess, and efficient preconditioning. Numerical results are presented to illustrate the application of the MNM to representative microwave circuit analysis problems.**

I. INTRODUCTION

The MoM formulation of Maxwell's equations leads to a dense system of complex equations, with thousands of unknowns when the geometry is electrically large, or when it has fine features that need to be resolved with sufficient accuracy. A direct solution of large equations using LU factorization is computer-intensive and this prompts one to turn to iterative solvers, such as those based on Krylov projection methods not only to alleviate the memory problem but to speed up the solution process as well. Typically, these iterative methods are used in conjunction with some type of preconditioner to improve convergence of the process [1-2]. Most of these Preconditioners provide satisfactory performance only when the geometry of the problem is electrically large; however, a typical MMIC structure is often only a fraction of the wavelength in dimensions, but still requires a large number of unknowns for accurate modeling. The matrices of these structures are often very poorly conditioned, and they do not exhibit the diagonally dominant behavior that characterizes a well-conditioned matrix.

In this paper, we introduce a novel approach referred to here in as the MNM technique, which improves the computational efficiency of iterative schemes used to solve problems of the type mentioned above, when the solution is desired over a range of frequencies, as is typically the case in microwave circuit analysis and CAD design. The MNM package is actually a combination of various schemes which work in consort with each other to enhance the performance of an iterative solver.

The first step in this process entails the equilibration of the coefficient matrix to reduce its condition number. Next, we apply a Multi-Frontal Preconditioner (MFP) to the equilibrated matrix to further decrease its condition number. Finally, we derive an initial guess, during the frequency sweeping process by following an algorithm described below, which improves the convergence of the iteration process further still.

A search through the literature reveals that numerous attempts have been made in the past [3] to derive a good guess for the solution via extrapolation (though not in the context of iteration), derived from the solutions at previous frequencies, but they have met with only limited success. The approach proposed herein involves an estimation of the solution vector based on the solutions at previous frequencies. The computational time involved in generating the estimate is negligible when compared to that of the MoM matrix generation and iterative solution.

We demonstrate the effectiveness of the proposed technique for improving the condition number *via* equilibration and using the initial guess generated by the MNM for preconditioned GMRES based iterative solution, by several examples. The paper demonstrates that the proposed technique helps improve the computational efficiency of the iterative solvers considerably, not only for MoM matrices associated with electrically large geometries, but also for poorly-conditioned matrices with a relatively small rank.

II. THE MNM ALGORITHM

In general, the conventional MoM formulation of the EM problem leads to a dense complex linear system of the form:

$$A X = B \quad (1)$$

where A is an N x N complex coefficient matrix, B and X are the known right hand side (RHS) and unknown vectors, respectively.

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The performance of any iterative solver is critically dependent on the condition number of the coefficient matrix A. It is imperative that the condition number of the matrix be improved before starting the iterative process. A three-step strategy has been adopted in this effort to improve the convergence behavior of the iterative solvers.

A. Equilibration

The first step is to equilibrate the matrix with a view to improving its condition number. Towards this end, we transform the original matrix A as follows:

$$Y = R\,A\,L \tag{2}$$

where R and L are the N x N right and the left equilibration diagonal matrices, and Y is the modified system matrix. The R and L are computed so that the maximum element in a given row or column of Y is unity. Using (2) in (1) yields

$$R^{-1}\,Y\,L^{-1}\,X = B \tag{3}$$

which is then rearranged as follows:

$$Y\,X' = R\,B \tag{4a}$$

$$X = L\,X' \tag{4b}$$

B. Preconditioning

Next, the modified system of linear equations given in (4a) is preconditioned by left-multiplying it with a matrix M as shown below:

$$M^{-1}Y\,X' = M^{-1}R\,B \tag{5}$$

The objective of this transformation is to improve the condition number of coefficient matrix, which, in turn, improves the convergence behavior of the iterative solver over that of the original system. Once the intermediate solution X' has been obtained, the actual solution X may be readily derived from (4b).

The preconditioning matrix M is derived by sparsifying the coefficient matrix Y row-wise. The individual elements of the resulting sparsified matrix M are generated as follows.

$$M_{i,j} = \begin{cases} Y_{i,j} & \text{if } \left|Y_{i,j}\right| > \alpha \left|Y_i^m\right| \\ 0 \end{cases} \tag{6}$$

where $\left|Y_i^m\right|$ is the magnitude of the largest element in the row 'i', and α is the threshold factor which is taken to be 10^{-4} for all the calculations presented in this paper. This form of sparsification results in a general non-zero pattern, which must be preserved in order for the solution procedure to be efficient. The MFP uses a combined unifrontal/multi-frontal approach to handle arbitrary sparsity patterns to achieve a general reduction in the fill-in [4]. The frontal approach involves finding a permutation of A, which when factorized into its LU factors (PAQ = LU where P and Q are permutation matrices), preserves the sparsity and the numerical accuracy.

C. Initial guess generation

Typically one is interested in investigating the performance of an MMIC design over a range of frequencies. We show how the solution at previous frequencies can be used to generate the initial estimate for the present frequency via an extrapolation procedure. The solutions at previous frequencies span a vector space $\mathbb{C}^m$ given by

$$\mathbb{C}^m \in \{x_{n-1}, x_{n-2}, \ldots, x_{n-m}\} \tag{7}$$

where the index 'n' corresponds to the current frequency at which the solution is desired, 'm' refers to the number of previous frequencies at which the solution is pre-computed and 'x' is the solution vector. The vectors belonging to $\mathbb{C}^m$ are next ortho-normalized by using a modified version of the Gram-Schmidt technique.

Let $\hat{x}_{n-i}$ be the i^{th} component of the modified vector space and let x_n be the solution to (1) that we desire to construct at the frequency 'n'. An estimate of this unknown vector may be generated by using a linear combination of the previous frequency solutions belonging to the modified $\mathbb{C}^m$ as follows.

$$x_n^{(0)} = \sum_{i=1}^{m} \alpha_i\,\hat{x}_{n-i} \tag{8}$$

where α's are the complex expansion coefficients, and $x_n^{(0)}$ is the initial estimate of the solution at the n^{th} frequency point. A possible solution for 'α' in (8) is then obtained by minimizing the projection of the $A\,x_i - B$ along each of the vectors belonging to the modified function space $\mathbb{C}^m$

III. Illustrative Results

The MNM approach, described in the previous section, has been applied to solve the MoM matrix equations generated for five test cases: (a) inter-digital filter at 4.5GHz; (b) two-layer coupled patch antenna at 1.8GHz; (c) edge-coupled microstrip filter at 9GHz; (d) a microstrip patch antenna with 7 radiating elements at 2.3GHz; and, (e) a microstrip patch antenna with 12 radiating elements at 2.3GHz. These test cases cover a wide spectrum of typical EM problems ranging from electrically-small MIC geometries (Cases -a and –c), to electrically-large geometries (Cases -b, -d, and -e), all of which lead to moderate-to-large size matrices.

TABLE I

EFFECT OF EQUILIBRATION ON THE CONDITION NUMBER OF MOM MATRIX

Case	# unknowns (N)	L_1 Condition number	
		Before EQU	After EQU
(a)	729	30 x 10^{11}	67 x 10^3
(b)	961	18 x 10^3	46 x 10^2
(c)	1093	36 x 10^4	44 x 10^3
(d)	2233	15 x 10^2	11 x 10^2
(e)	3828	16 x 10^2	12 x 10^2

For Cases (a) and (c), a large number of unknowns are required in the formulation of the problem because of a very fine discretization, which is of the order of $\lambda/169$ in some regions. The number of unknowns in the test cases varied from 729 to 3828. The condition number of the MoM matrix for each of the above 5 test cases are presented in Table.1, before and after equilibration. The fine geometry discritization employed in the Cases (a) and (c) resulted in a highly ill-conditioned MoM matrix with a condition number of $30\text{x}10^{11}$ and $36\text{x}10^{4}$, respectively, while this number was more reasonable ($\sim 10^3$ or less) for the rest of the examples. However, as seen from Table.1, the equilibration process improved the condition number considerably in all of these cases, and this effect was most pronounced when the original matrix was highly ill-conditioned.

Next, we examine the eigenspectrum of the MoM matrix for Case-a before equilibration, and plot it in Fig.1a. It is evident that the poor conditioning of the matrix results from the presence of very small eigenvalues close to zero. This is not entirely unexpected, since a fine discretization renders some of the rows and columns of the original matrix closely dependent, and thereby reduces its rank. The eigen spectrum of $M^{-1}Y$ is presented in Fig.1b to illustrate the fact that the equilibration together with the MFP, described in the previous section, improves the conditioning. It can be seen from Fig.1b that in contrast to the widely spread spectrum of Fig.1a, the eigenvalues of the $M^{-1}Y$ are clustered around unity.

The performances of iterative solvers CGNR and GMRES, have been studied for the five test cases mentioned above, and the results are presented in Table.2. The solution time for the LAPACK direct solver using complete LU decomposition is also included in this table for comparison. All the computations have been carried out on a Pentium III 662MHz Xeon PC with a 512Mb RAM, and a residual error of 0.001 has been chosen as the convergence criterion for the iterative solvers. The CPU time shown for the iterative solvers includes the time taken for equilibration, generation of the preconditioner matrix, as well as that needed by the iterative solver itself.

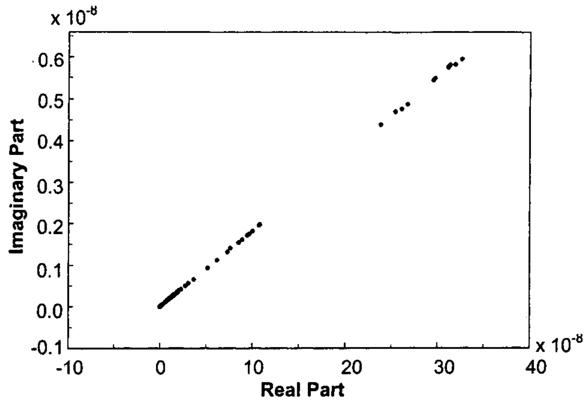

Fig.1a. Eigenspectrum of the original MoM matrix of Inter-digital filter (Case-a) with 729 unknowns.

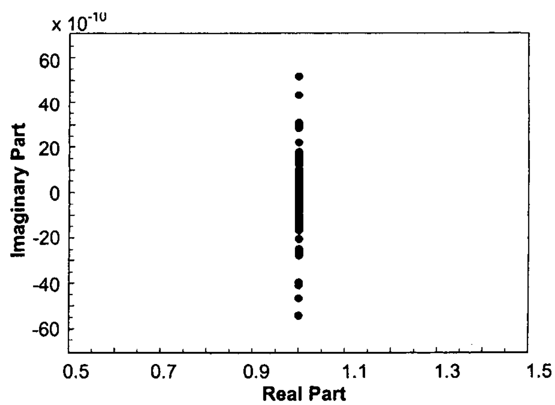

Fig.1b. Eigen spectrum of the original MoM matrix of Inter-digital filter (Case-a) with 729 unknowns after equilibration and preconditioning.

The initial guess generation procedure for the iterative solver is demonstrated by analyzing a 8 element microstrip patch array antenna over a range of frequencies that includes the patch resonance frequency. The number of iterations required in order to achieve a residual error 1% in the solution is presented as a function of frequency in

TABLE II
COMPARISON OF THE PERFORMANCE OF VARIOUS SOLVERS

Case	Direct Solution time (s)	CGNR solver # iterations	CGNR solver solution time (s)	GMRES solver # iterations	GMRES solver solution time (s)
(a)	4.70	9	3.40	4	1.37
(b)	10.70	6	5.23	5	3.40
(c)	15.85	12	11.32	5	4.82
(d)	248.84	7	19.36	6	10.11
(e)	1268.55	6	52.58	8	38.53

Fig.2 for this case. The number of iterations needed for the iterative solver with the zero initial guess is also presented for the sake of comparison. The frequency step is taken to be 10MHz, and a 2-vector MNM has been used. It can be seen that the iterative solver requires less number of iterations when using the extrapolated initial guess over the entire band except near the resonant frequency of 2.3GHz. The total CPU time taken by the solver with zero initial guess is 962s, while it is 730s when using the extrapolated initial guess.

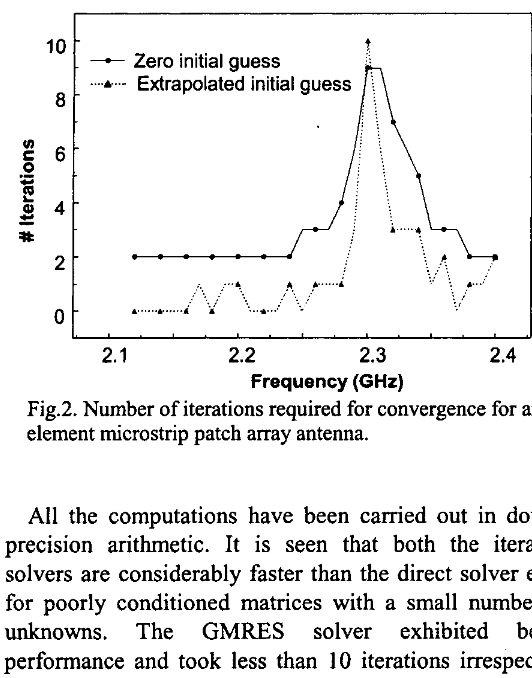

Fig.2. Number of iterations required for convergence for an 8 element microstrip patch array antenna.

All the computations have been carried out in double precision arithmetic. It is seen that both the iterative solvers are considerably faster than the direct solver even for poorly conditioned matrices with a small number of unknowns. The GMRES solver exhibited better performance and took less than 10 iterations irrespective of the number of unknowns. For a comparable number of iterations (Cases -b and -d), the CGNR solver requires nearly twice the CPU time when compared to the GMRES solver due to the additional operations involving the adjoint operator.

IV. CONCLUSION

An efficient approach has been presented for the solution of large dense system of linear equations arising in the integral equation formulation of electromagnetic problems. A three-step process has been introduced in which the condition number of the matrix is first improved by equilibration, and then further enhanced by preconditioning. The initial guess for the iterative solution is generated by using an extrapolation technique. It has been shown that the proposed approach considerably improves the computational efficiency of the iterative solvers, e.g., CGNR and GMRES, even for poorly conditioned MoM matrices with a relatively small number of unknowns.

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